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Operator product expansion of higher rank Wilson loops from D-branes and matrix models

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Abstract: In this paper we study correlation functions of circular Wilson loops in higher dimensional representations with chiral primary operators of $\mathcal{N} = 4$ super Yang-Mills theory. This is done using the recently established relation between higher rank Wilson loops in gauge theory and D-branes with electric fluxes in supergravity. We verify our results with a matrix model computation, finding perfect agreement in both the symmetric and the antisymmetric case.

Keywords: D-branes, AdS-CFT Correspondence, Matrix Models.

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1. Introduction

In the AdS/CFT correspondence local gauge invariant operators are matched with bulk supergravity fields evaluated at the boundary of the AdS space [1−3]. An important role in the correspondence is also played by non local gauge invariant operators, the most notable example being the Wilson loop.

In $\mathcal{N} = 4$ super Yang-Mills theory Wilson loops are defined (in Euclidean signature) as

$$W_{\mathcal{R}}(\mathcal{C}) = \frac{1}{\dim\mathcal{R}} \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \oint_{\mathcal{C}} d\tau \left[i A_\mu(\tau) \dot{x}^\mu + \Phi_I(\tau) \theta^I |\dot{x}| \right] \tag{1.1}$$

where A_μ is the gauge field and Φ_I are the six scalars of the $\mathcal{N} = 4$ multiplet, and θ^I is a constant unit vector in $\mathbb{R}^6$. The data which characterize the Wilson loop are the shape of the integration contour $\mathcal{C}$ and the representation $\mathcal{R}$ of the gauge group. Supersymmetry restricts

$\mathcal{C}$ to be a straight line or a circle [4],[1] while $\mathcal{R}$ may be arbitrary. It is a well-established part of the AdS/CFT correspondence that Wilson loops in the fundamental representation are associated with a classical string surface of minimal area landing on the loop [7, 8]. The worldsheet area, being the string infinitely long, is formally infinite but the string action can nevertheless be made finite by adding suitable counterterms [9]. The expectation value of the Wilson loop is then the partition function of the regularized string associated to it.

Recently circular Wilson loops in representations other than the fundamental have been a very active field of investigation. A holographic dictionary has emerged, where probe branes in the bulk are related to higher rank Wilson loops in the boundary [12–15].[2] In particular, D3 branes with $AdS_2 \times S^2$ worldvolume and k units of fundamental string charge dissolved on them have been proved to compute expectation values of Wilson loops in the rank k symmetric representation.[3] This picture is the natural generalization to the $AdS_5 \times S^5$ background of the idea that a fundamental string ending on a D3 brane in flat space can be described in terms of a curved D3 brane with a localized spike carrying a unit of electric flux, as first proposed by Callan and Maldacena [18]. On the other hand, D5 branes with $AdS_2 \times S^4$ worldvolume and k units of string charge correspond to Wilson loops in the rank k antisymmetric representation. Both these branes are half-BPS and preserve the same isometries of (1.1), namely $\mathrm{SO}(2,1) \times \mathrm{SO}(3) \times \mathrm{SO}(5)$. They pinch off at the boundary of AdS_5 landing on the curve that defines the Wilson loop.

The intuitive reason for considering these objects is that, to build a Wilson loop in the rank k representation, one would start with considering k coincident fundamental strings. The $\mathrm{D3}_k$ and $\mathrm{D5}_k$ branes can then be thought of as coming from an Emparan-Myers polarization effect [19, 20], which, for k sufficiently large, blows up a $S^2 \subset AdS_5$ or a $S^4 \subset S^5$ from the worldsheet of the k coincident strings. This is reminiscent of the interpretation of gravitons with large momenta as D3 branes wrapping a $S^3 \subset S^5$ (giant gravitons) or a $S^3 \subset AdS_5$ (dual giant gravitons). We can then regard the $\mathrm{D3}_k$ and $\mathrm{D5}_k$ branes as dual giant and giant Wilson loops, respectively.

This brane picture has the advantage of automatically encoding the interactions between the coincident strings [21] and yields all non planar contributions to the expectation value of the higher rank Wilson loop [12].

It is well-known that the expectation value of circular Wilson loops in the fundamental representation can be computed with a quadratic Hermitian matrix model [22, 23]. It has been conjectured that this can be extended to higher rank loops and matrix model computations have provided a successful check of the holographic dictionary just discussed [12, 24, 25].

[1] In this paper we consider only half-BPS operators. For loops preserving less supersymmetry see [4–6].

[2] For 't Hooft loops see [12, 16], and for applications to finite temperature see [17].

[3] The brane probe approximation breaks down for k much larger than N. In this limit the branes backreact deforming the geometry into the supergravity solutions studied in [10] and [11].

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A small circular Wilson loop, when probed from a distance much larger than its characteristic size, can be expanded in a series of local operators of different conformal dimension [26]. The operators which are allowed to appear in the expansions must preserve the same symmetries of (1.1) and therefore must be bosonic, gauge invariant and SO(5) invariant. The conformal dimension of some of these operators is not protected by the superconformal algebra and therefore they receive large anomalous dimensions and decouple in the strong coupling regime. An important class of operators which have protected dimensions and appear in the operator product expansion are the chiral primary operators. The correlator with a local operator can then be read off from the expansion of the Wilson loop [26].[4]

In this paper, we use the $\mathrm{D3}_k$ and $\mathrm{D5}_k$ branes to compute the correlation function between a circular Wilson loop in a higher representation and a chiral primary operator in the fundamental representation. We do this by studying the coupling to the brane worldvolume of the supergravity modes dual to the chiral primaries. These modes propagate from the insertion of the local operator on the boundary to the brane worldvolume in the bulk.

The paper is organized as follows. In sections 2 and 3 we review the operator product expansion of the Wilson loop and how to compute correlation functions when the Wilson loop is described in terms of a fundamental string worldsheet. To evaluate this one needs to study the harmonic expansion on S^5 of the bulk fields which couple to the worldsheet.

Following the philosophy outlined before, we then replace the fundamental string with the $\mathrm{D3}_k$ and $\mathrm{D5}_k$ branes. We start by investigating the symmetric case in section 4. We expand the brane action to linear order in the fluctuations of the bulk fields and find how it couples to the relevant supergravity modes. Using the procedure reviewed in section 3 we compute the correlation function. In the limit of small k we recover the previously known result derived using the fundamental string.

We then move on to the analysis of the antisymmetric case. The $\mathrm{D5}_k$ brane now extends also in the S^5 directions. Also in this case we compute the correlator between the Wilson loop and a chiral primary operator and check that it yields the correct string limit.

As a further check, in section 5, we compare our results against the expressions coming from the normal matrix model introduced in this context in [24] and find perfect agreement both in the symmetric and antisymmetric case.

In the appendix we collect some facts about spherical harmonics and orthogonal polynomials that we have used in the paper.

2. Kaluza-Klein expansion

In this section we review the expansion in spherical harmonics for type IIB supergravity on

[4] For a nice review see [27].

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$AdS_5 \times S^5$ [28], and identify the bulk excitations associated to turning on a chiral primary operator in the dual $\mathcal{N}=4$ gauge theory [29]. These will be later used to construct the coupling of the various supergravity modes to the D3_k and D5_k branes.

The Einstein equations read [5]

$$R_{mn} = \frac{1}{96} F_{mijkl} F_n{}^{ijkl} \tag{2.1}$$

where the 5-form field strength $F_{(5)}$ is self-dual. In the Poincare patch, the $AdS_5 \times S^5$ solution reads

$$ds^2 = \frac{1}{z^2}\left(dz^2 + d\vec{x}^{\,2}\right) + d\Omega_5^2 \tag{2.2}$$
$$\bar{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = -4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \qquad \bar{F}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = -4\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}. \tag{2.3}$$

The fluctuations around the background geometry can be parametrized as follows

$$G_{mn} = g_{mn} + h_{mn} \tag{2.4}$$
$$h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{h_2}{5} g_{\alpha\beta}, \qquad g^{\alpha\beta} h_{(\alpha\beta)} = 0 \tag{2.5}$$
$$h_{\mu\nu} = h'_{\mu\nu} - \frac{h_2}{3} g_{\mu\nu}, \qquad g^{\mu\nu} h'_{(\mu\nu)} = 0 \tag{2.6}$$
$$F = \bar{F} + \delta F, \qquad \delta F_{ijklm} = 5\nabla_{[i} a_{jklm]} \tag{2.7}$$

where h_2 is the trace of the metric on the five-sphere, $h_2 \equiv h_{\alpha\beta} g^{\alpha\beta}$. Note that the fields $h_{\mu\nu}$ and $h'_{\mu\nu}$ are related by a $d=5$ Weyl shift. To identify the bulk excitation in AdS_5 we expand the fluctuations as follows [6]

$$h'_{\mu\nu} = \sum h'^{I}_{\mu\nu}(x) Y^I(y) \tag{2.8}$$
$$h_2 = \sum h_2^I(x) Y^I(y) \tag{2.9}$$
$$a_{\mu_1\mu_2\mu_3\mu_4} = \sum a^I_{\mu_1\mu_2\mu_3\mu_4}(x) Y^I(y) \tag{2.10}$$
$$a_{\alpha_1\alpha_2\alpha_3\alpha_4} = -4\sum \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^I(x) \nabla^\alpha Y^I(y) \tag{2.11}$$

where x and y refer to the AdS_5 and S^5 coordinates respectively, and Y^I are scalar spherical harmonics on the 5-sphere which satisfy [7]

$$\nabla^\alpha \nabla_\alpha Y^I = -\Delta(\Delta+4) Y^I. \tag{2.12}$$

Spherical harmonics on S^5 can be classified in terms of the SO(6) $\simeq$ SU(4) R-symmetry group. In particular scalar harmonics belong to the $[0,\Delta,0]$ representation. The fields

[5] In our conventions Latin indices run over the whole 10 dimensional manifold while Greek indices $\mu,\nu,\ldots$ and $\alpha,\beta,\ldots$ run over AdS_5 and S^5 respectively. We also choose units in which $R_{AdS_5} = R_{S^5} = 1$.

[6] We do not consider the harmonic expansion of $h_{(\alpha\beta)}$ as this fluctuation is related to $Q^2\bar{Q}^2$ descendants of chiral primaries in the dual super Yang-Mills theory.

[7] We include a brief review of spherical harmonics in the appendix.

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h_2 and b appear coupled in the linearized equation of motions. Their equations can be diagonalized introducing the linear combinations [29]

$$s^I = \frac{1}{20(\Delta+2)}[h_2^I - 10(\Delta+4)b^I] \tag{2.13}$$

$$t^I = \frac{1}{20(\Delta+2)}[h_2^I + 10\Delta b^I] \tag{2.14}$$

which obey the equations of motion

$$\nabla_\mu \nabla^\mu s^I = \Delta(\Delta-4)\, s^I \tag{2.15}$$

$$\nabla_\mu \nabla^\mu t^I = (\Delta+4)(\Delta+8)\, t^I . \tag{2.16}$$

A scalar field in AdS with $m^2 = \Delta(\Delta-4)$ (with $\Delta \geq 2$) transforming in the $[0,\Delta,0]$ representation corresponds to a chiral primary operator $\mathcal{O}_\Delta$ of conformal dimension Δ. Therefore, to linear order, the scalar field s^I corresponds to chiral primaries in the dual gauge theory. On the other hand, the scalars t^I are associated to their descendants, which we do not consider in the paper.

The linear solutions to the equations of motion turn out to be [29]

$$h_{\mu\nu} = -\frac{6}{5}\Delta\, s\, g_{\mu\nu} + \frac{4}{\Delta+1}\nabla_{(\mu}\nabla_{\nu)} s \tag{2.17}$$

$$h_{\alpha\beta} = 2\Delta\, s\, g_{\alpha\beta} \tag{2.18}$$

$$a_{\mu_1\mu_2\mu_3\mu_4} = 4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5} b \tag{2.19}$$

$$a_{\alpha_1\alpha_2\alpha_3\alpha_4} = -4\sum_I \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^I(x)\nabla^\alpha Y^I(y) \tag{2.20}$$

where $s = \sum s^I Y^I$ and $b = \sum b^I Y^I$. Using (2.5) and the solution (2.18) one can identify $h_2 = 10\,\Delta\, s$. Setting $t^I = 0$ in (2.14), one can then deduce $s = -b$.

3. Operator product expansion of Wilson loops

The Wilson loop operator can be expanded in terms of local operators when probed from distances much larger than its characteristic size a. For the circular Wilson loop with radius a we can write [26]

$$W(\mathcal{C}) = \langle W(\mathcal{C})\rangle \left(1 + \sum_n c_{(n)} a^{\Delta_{(n)}} \mathcal{O}_{(n)}\right) . \tag{3.1}$$

In this expression $\mathcal{O}_{(n)}$ is a local gauge invariant operator with conformal dimension $\Delta_{(n)}$, and the sum over n runs over both the primary operators and their descendants. This operator product expansion must be invariant under the symmetries preserved by the Wilson loop. The half-BPS circular loop has $\theta^I(\tau) = \theta^I = const.$ and therefore preserves a SO(5) subgroup of the original SO(6) R-symmetry group. The operators appearing in the

OPE expansion must therefore contain SO(5) singlets in the SO(6) $\to$ SO(5) decomposition. For example, at level $\Delta = 2$ we can consider the chiral primary operator $\mathcal{O}_2^A = C_{IJ}^A \mathrm{Tr}\Phi^I\Phi^J$, where C_{IJ}^A is a SO(6) symmetric traceless tensor. Under SO(6) $\to$ SO(5) it decomposes as $\mathbf{20} \to \mathbf{1} + \mathbf{5} + \mathbf{14}$ and therefore, containing a singlet, it will appear in the OPE of the Wilson operator. A similar analysis can be performed for higher dimension operators, which in general will contain covariant derivatives, gauge field-strenghts and the fermions of the $\mathcal{N} = 4$ multiplet. Some of them will get large anomalous dimension in the strong coupling limit and therefore will decouple. The generic expansion looks as follows

$$\begin{aligned}\frac{W(\mathcal{C})}{\langle W(\mathcal{C})\rangle} &= 1 + c_{(2)}\, a^2\, Y_A^{(2)}(\theta)\mathcal{N}_2 C_{IJ}^A \mathrm{Tr}\left(\Phi^I\Phi^J\right) + \\ &\quad + c_{(3)}\, a^3\, Y_A^{(3)}(\theta)\mathcal{N}_3 C_{IJK}^A \mathrm{Tr}\left(\Phi^I\Phi^J\Phi^K\right) + c_{(4)}\, a^3 \mathrm{Tr}\big(\theta^I X^I F_+\big) + \cdots\end{aligned} \tag{3.2}$$

where $Y_A^{(n)}(\theta)$ are spherical harmonics and $\mathcal{N}_n$ are normalization constants.

The coefficients appearing in the OPE expansion can be read off from the large distance behavior of the two point correlator of the Wilson loop and the local operators

$$\frac{\langle W(\mathcal{C})\mathcal{O}^{(n)}(x)\rangle}{\langle W(\mathcal{C})\rangle} = c_{(n)}\frac{a^{\Delta_{(n)}}}{L^{2\Delta_{(n)}}} + \cdots \tag{3.3}$$

where it is assumed that the loop radius a is much smaller than the distance L from the point of insertion of the local operator. In this paper we will focus only on chiral primaries operators $\mathcal{O}_\Delta^A = C_{I_1\cdots I_\Delta}^A \mathrm{Tr}(\Phi^{I_1}\dots\Phi^{I_\Delta})$.[8] These belong to short representations of the superconformal algebra, have protected conformal dimensions, and appear at all orders in the expansion (3.2).

In the AdS/CFT correspondence the chiral primary operators are dual to supergravity modes: $\mathcal{O}_\Delta$ corresponds to a scalar of mass $m^2 = \Delta(\Delta - 4)$, which is a combination of the trace of the metric and the RR 4-form over S^5, as we reviewed in the previous section.

We now briefly discuss the procedure for computing the correlation function of these operators with a Wilson loop in the strong coupling regime. The coupling to the string worldsheet of the supergravity mode dual to $\mathcal{O}_\Delta$ is given by a vertex operator V_Δ, which can be determined by expanding the string action to linear order in the fluctuation $h_{\mu\nu}$

$$\begin{aligned}S &= \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\det\left(G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\right)} \\ &= \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\det\left(g_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\right)}\left(1 + \frac{1}{2}(g_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu)^{-1}h_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu + \cdots\right).\end{aligned} \tag{3.4}$$

The fluctuation of the metric $h_{\mu\nu}$ on AdS_5 is given in eq. (2.17). We write the scalar s^I in terms of a source s_0^I located at the boundary

$$s^I(\vec{x}, z) = \int d^4\vec{x}\,' G_\Delta(\vec{x}\,'; \vec{x}, z) s_0^I(\vec{x}\,') \tag{3.5}$$

[8] We take the traces of the chiral primaries in the fundamental representation.

where $G_\Delta(\vec{x}\,';\vec{x},z)$ is the bulk-to-boundary propagator which describes the propagation of the supergravity mode from the insertion point $\vec{x}\,'$ of the chiral primary operator to the point $(\vec{x},z)$ on the string worldsheet

$$G_\Delta(\vec{x}\,';\vec{x},z) = c\left(\frac{z}{z^2+|\vec{x}-\vec{x}\,'|^2}\right)^\Delta . \tag{3.6}$$

The constant $c = \frac{\Delta+1}{2^{2-\Delta/2}N\sqrt{\Delta}}$ is fixed by requiring the unit normalization of the 2-point function [26]. Since we are probing the Wilson loop from a distance L much larger than its radius a we can approximate

$$G_\Delta(\vec{x}\,';\vec{x},z) \simeq c\,\frac{z^\Delta}{L^{2\Delta}}\,, \qquad \partial_z s^I \simeq \frac{\Delta}{z}s^I\,, \qquad \partial_z^2 s^I \simeq \frac{\Delta(\Delta-1)}{z^2}s^I. \tag{3.7}$$

The relevant Christoffel symbols are readily computed to be

$$\Gamma^z_{\mu\nu} = z g_{\mu\nu} - \frac{2}{z}\delta^z_\mu\delta^z_\nu \tag{3.8}$$

so that one finally has

$$h^I_{\mu\nu} \simeq -2\Delta\; g_{\mu\nu}s^I + \frac{4\Delta}{z^2}\delta^z_\mu\delta^z_\nu s^I. \tag{3.9}$$

Inserting this result into (3.4), the coupling to the worldsheet is found to be [26]

$$\frac{1}{2\pi\alpha'}\int d\mathcal{A}\,(-2\Delta\, s)\frac{z^2}{a^2} \equiv \frac{1}{2\pi\alpha'}\int d\mathcal{A}\,V_\Delta\, s. \tag{3.10}$$

In this expression $d\mathcal{A}$ is the area element of the classical string. The correlation function is obtained from functionally differentiating the previous formula with respect to the source s_0

$$\begin{aligned}\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(\vec{x}_0)\rangle}{\langle W(\mathcal{C})\rangle} &= -Y^I(\theta)\frac{\delta}{\delta s_0(\vec{x}_0)}\frac{1}{2\pi\alpha'}\int d\mathcal{A}\,d^4x'\,V_\Delta\,G_\Delta(\vec{x}\,';\vec{x},z)s_0^I(\vec{x}\,')\\ &= -Y^I(\theta)\frac{1}{2\pi\alpha'}\int d\mathcal{A}\;V^\Delta G_\Delta(\vec{x}_0;\vec{x},z).\end{aligned} \tag{3.11}$$

One obtains in the approximations of eq. (3.7)

$$\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(\vec{x}_0)\rangle}{\langle W(\mathcal{C})\rangle} = 2^{\Delta/2-1}\frac{\sqrt{\Delta\lambda}}{N}\frac{a^\Delta}{L^{2\Delta}}. \tag{3.12}$$

We now move on to studying the operator product expansion of Wilson loops in higher dimensional representations. We analyze the rank k symmetric representation first. In the bulk this is described by a D3$_k$ brane.

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4. Brane computation

4.1 The D3 brane

We consider a small circular Wilson loop of radius a placed on the boundary of AdS_5. The metric on AdS_5 can be written in polar coordinates as

$$ds^2_{\text{AdS}} = \frac{1}{z^2}\left(dz^2 + dr_1^2 + r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2\right). \tag{4.1}$$

The position of the loop is defined by $r_1 = a$ and $z = r_2 = 0$. We take a D3 brane which pinches off on this circle as $z \to 0$ and preserves a $\mathrm{SO}(2,1) \times \mathrm{SO}(3) \times \mathrm{SO}(5)$ isometry [12].

The bulk action includes a DBI part and a Wess-Zumino term, which captures the coupling of the background Ramond-Ramond field to the brane

$$S_{D3} = T_{D3}\int \sqrt{\det(\gamma + 2\pi\alpha' F)} - T_{D3}\int P[C_{(4)}] \tag{4.2}$$

where $T_{D3} = \frac{N}{2\pi^2}$ is the tension of the brane, γ is the induced metric, F the electromagnetic field strenght, and $P[C_{(4)}]$ is the pull-back of the 4-form

$$C_{(4)} = \frac{r_1 r_2}{z^4} dr_1 \wedge d\psi \wedge dr_2 \wedge d\phi \tag{4.3}$$

to the brane worldvolume.

We review the brane solution found in [12]. It turns out to be more convenient to use a new set of coordinates obtained by transforming $\{z,\, r_1,\, r_2\}$ into

$$z = \frac{a\sin\eta}{\cosh\rho - \sinh\rho\cos\theta}\,, \quad r_1 = \frac{a\cos\eta}{\cosh\rho - \sinh\rho\cos\theta}\,, \quad r_2 = \frac{a\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta}. \tag{4.4}$$

In this coordinate system the metric on AdS_5 reads

$$ds^2_{\text{AdS}} = \frac{1}{\sin^2\eta}\left(d\eta^2 + \cos^2\eta\, d\psi^2 + d\rho^2 + \sinh^2\rho\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right) \tag{4.5}$$

where $\rho \in [0,\infty)$, $\theta \in [0,\pi]$, and $\eta \in [0,\pi/2]$. The Wilson loop is located at $\eta = \rho = 0$. One can pick a static gauge in which the worldvolume coordinates of the brane are identified with $\{\psi\,, \rho\,, \theta\,, \phi\}$ and the brane sits at a fixed point of the S^5 determined by the constant unit vector $\theta^I \in \mathbb{R}^6$. The remaining coordinate can be seen as a scalar field, $\eta = \eta(\rho)$. Because of the symmetries of the problem the electromagnetic field has only one component, $F_{\psi\rho}(\rho)$. In this coordinates the DBI action in (4.2) reads

$$S_{DBI} = 2N\int d\rho d\theta\, \frac{\sin\theta\sinh^2\rho}{\sin^4\eta}\sqrt{\cos^2\eta(1+\eta'^2) + (2\pi\alpha')^2\sin^4\eta F^2_{\psi\rho}} \tag{4.6}$$

while the Wess-Zumino term is

$$S_{WZ} = -2N\int d\rho d\theta\, \frac{\cos\eta\sin\theta\sinh^2\rho}{\sin^4\eta}\left(\cos\eta + \eta'\sin\eta\,\frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta}\right). \tag{4.7}$$

The solution to the equations of motion reads [12]

$$\sin\eta = \frac{1}{\kappa}\sinh\rho\,, \qquad F_{\psi\rho} = \frac{ik\lambda}{8\pi N\sinh^2\rho}\,, \qquad \kappa = \frac{k\sqrt{\lambda}}{4N}. \tag{4.8}$$

The bulk action has to be complemented with boundary terms for the worldvolume scalar η and for the electric field $F_{\psi\rho}$ [12]. These terms do not change the solution but alter the final value of the on-shell action which reads

$$S_{D3} = S_{DBI} + S_{WZ} + S_{\rm bdy} = -2N(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa). \tag{4.9}$$

The expectation value of a Wilson loop in the rank k symmetric [9] representation is then

$$\langle W_{S_k}\rangle = \exp\left(2N(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa)\right). \tag{4.10}$$

For small κ this expression reproduces the result of k fundamental strings

$$\langle W_{S_k}\rangle \simeq e^{k\sqrt{\lambda}}. \tag{4.11}$$

4.1.1 Coupling to chiral primaries

The linearized coupling of the scalar s^I to the brane worldvolume can be found by expanding the induced metric on the brane around the $AdS_5\times S^5$ background g_{mn} and keeping the first order term in the fluctuation h_{mn}. Since the brane lies completely in AdS_5 we can write

$$\begin{aligned} S_{DBI} &= T_{D3}\int d^4\sigma\sqrt{\det\left(G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + 2\pi\alpha' F_{ab}\right)} \\ &= T_{D3}\int d^4\sigma\sqrt{\det\left(g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + 2\pi\alpha' F_{ab}\right)}\,\cdot \\ &\qquad\cdot\left(1+\frac{1}{2}\left(g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + 2\pi\alpha' F_{ab}\right)^{-1} h_{\rho\sigma}\partial_a X^\rho\partial_b X^\sigma + \cdots\right). \end{aligned} \tag{4.12}$$

Here a, b are the brane worldvolume indices.

The coupling of s^I to the 4-form in the Wess-Zumino term is obtained by replacing $C_{(4)}\to C_{(4)}+a_{(4)}$ where, using eq. (2.19) and the approximation (3.7), the fluctuation $a_{(4)}$ is

$$a_{\mu_1\ldots\mu_4} \simeq -4\epsilon_{\mu_1\ldots\mu_4 z}\partial^z s^I \simeq -4\Delta\, z\,\epsilon_{\mu_1\ldots\mu_4 z}s^I \tag{4.13}$$

so that

$$S^{(1)}_{WZ} = -T_{D3}\int P[a_{(4)}] = 4\,T_{D3}\,\Delta\int P[C_{(4)}]s^I. \tag{4.14}$$

[9] In the limit of $N\to\infty$ and $\lambda\to\infty$ the symmetric representation coincides with the multiply wound Wilson loop [24, 25].

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We use now the explicit solution to the equations of motion (4.8) to evaluate the on-shell value of the fluctuations (4.12) and (4.14). The first order in the fluctuation in (4.12) turns out to be

$$S^{(1)}_{DBI} = 4N\Delta\kappa^2 \int d\rho d\theta \, \frac{\sin\theta}{\sinh^2\rho} \left(-1 - 2\kappa^2 + \frac{1 - \sinh^2\rho(\kappa^{-2} - \sin^2\theta)}{(\cosh\rho - \sinh\rho\cos\theta)^2} \right) s^I. \tag{4.15}$$

Similarly, the Wess-Zumino term reads

$$S^{(1)}_{WZ} = 8N\Delta\kappa^4 \int d\rho d\theta \, \frac{\sin\theta}{\sinh^2\rho} \left(1 + \frac{1}{\kappa^2} \frac{\sinh^3\rho - \sinh\rho\cosh^2\rho}{\cosh\rho - \sinh\rho\cos\theta} \cos\theta \right) s^I. \tag{4.16}$$

The final result for the action is then

$$S^{(1)}_{D3} = S^{(1)}_{DBI} + S^{(1)}_{WZ} = -4N\Delta \int_0^{\sinh^{-1}\kappa} d\rho \int_0^{\pi} d\theta \, \frac{\sin\theta}{(\cosh\rho - \sinh\rho\cos\theta)^2} s^I. \tag{4.17}$$

4.1.2 The correlation function

The prescription for computing the correlation function between the Wilson loop and the chiral primary operator is to functionally differentiate the action (4.17) with respect to the source s^I_0 (see eq. (3.5))

$$\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(L)\rangle}{\langle W(\mathcal{C})\rangle} = -\frac{\delta S^{(1)}_{D3}}{\delta s_0}\bigg|_{s_0=0}. \tag{4.18}$$

We approximate the bulk-to-boundary propagator with $c\frac{z^\Delta}{L^{2\Delta}}$ and use for z the expression (4.4). This yields

$$\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(L)\rangle}{\langle W(\mathcal{C})\rangle} \simeq \frac{a^\Delta}{L^{2\Delta}} \frac{4N\Delta}{\kappa^\Delta} c \int_0^{\sinh^{-1}\kappa} d\rho \sinh^\Delta\rho \int_0^{\pi} d\theta \frac{\sin\theta}{(\cosh\rho - \sinh\rho\cos\theta)^{2+\Delta}}. \tag{4.19}$$

We are neglecting terms of higher order in $\frac{a}{L^2}$.

After performing the two integrals, the final result for the coefficients of the operator product expansion turns out to be remarkably simple

$$c_{S_k,\Delta} = \frac{2^{\Delta/2+1}}{\sqrt{\Delta}} \sinh(\Delta \sinh^{-1}\kappa). \tag{4.20}$$

Interestingly enough, this can be expressed in terms of Chebyshev polynomials with imaginary argument

$$c_{S_k,\Delta} = \frac{(-1)^{\Delta/2} 2^{\Delta/2+1}}{\sqrt{\Delta}} \cdot \begin{cases} -iV_\Delta(i\kappa) & \text{for } \Delta \text{ even} \\ T_\Delta(i\kappa) & \text{for } \Delta \text{ odd} \end{cases} \tag{4.21}$$

where we have used the identities $T_n(x) = \cos(n\cos^{-1}x)$ and $V_n(x) = \sin(n\cos^{-1}x)$.

The string limit is recovered when $\kappa \to 0$. In this regime the S^2 in the brane worldvolume shrinks to zero size and the D3 reduces effectively to a fundamental string with AdS_2 worldsheet. The coefficients (4.20) become

$$c_{S_k,\Delta} \simeq 2^{\Delta/2+1}\sqrt{\Delta}\,\kappa = 2^{\Delta/2-1}\frac{\sqrt{\Delta\lambda}}{N}\,k \tag{4.22}$$

in perfect agreement with the result (3.12) found originally in [26].

4.2 The D5 brane

Circular Wilson loops in the rank k antisymmetric representation of the gauge group have a bulk description in terms of D5 branes with $AdS_2 \times S^4$ worldvolume and k units of fundamental string charge dissolved in them [13, 14]. The $D5$ description of these Wilson loops is valid in the large N, large λ limit with k/N fixed. Before moving on to compute the coupling of these branes to the scalars s^I dual to chiral primaries, we briefly review the D5 solution [13] to set up the notation and our conventions. It is convenient to take the $AdS_5 \times S^5$ metric as

$$\begin{aligned} ds^2 &= \cosh^2 u (d\zeta^2 + \sinh^2\zeta d\psi^2) + du^2 + \sinh^2 u (d\vartheta^2 + \sin^2\vartheta d\phi^2) + \\ &\quad + d\theta^2 + \sin^2\theta d\Omega_4^2 \,, \end{aligned} \tag{4.23}$$

where we have written the AdS_5 factor as an $AdS_2 \times S^2$ fibration. These coordinates are related to the usual Poincare patch by

$$\begin{aligned} r_1 &= \frac{a \cosh u \sinh\zeta}{\cosh u \cosh\zeta - \cos\vartheta \sinh u} \,, \qquad r_2 = \frac{a \sinh u \sin\vartheta}{\cosh u \cosh\zeta - \cos\vartheta \sinh u} \\ z &= \frac{a}{\cosh u \cosh\zeta - \cos\vartheta \sinh u} \end{aligned} \tag{4.24}$$

where, as before, a denotes the radius of the Wilson loop. In these coordinates, the Wilson loop is at $\zeta \to \infty$, $u = 0$ and it is parametrized by ψ. The selfdual 4-form potential can be taken to be

$$C_{(4)} = 4\left(\frac{u}{8} - \frac{1}{32}\sinh 4u\right) dH_2 \wedge d\Omega_2 - \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right) d\Omega_4 \,, \tag{4.25}$$

where dH_2 denotes the volume element of the AdS_2 part of the metric.

Since we want to construct a $D5$ brane with $AdS_2 \times S^4$ worldvolume, it is natural to take a static gauge in which ζ, ψ and the coordinates of the $S^4 \subset S^5$ are the worldvolume coordinates. Furthermore we can take the following ansatz which preserves the $\mathrm{SO}(2,1) \times \mathrm{SO}(3) \times \mathrm{SO}(5)$ symmetry of the Wilson loop

$$u = 0 \,, \qquad \theta = const. \tag{4.26}$$

and only the $F_{\psi\zeta}$ component of the worldvolume gauge field is turned on. With this ansatz the DBI and Wess-Zumino parts of the D5 action reduce to

$$\begin{aligned} S_{DBI} &= T_{D5} \int d^6\sigma \sqrt{\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \\ &= \frac{2N}{3\pi}\sqrt{\lambda} \int d\zeta \sinh\zeta \sin^4\theta \sqrt{1 + \frac{4\pi^2}{\lambda}\frac{F_{\psi\zeta}^2}{\sinh^2\zeta}} \,, \end{aligned} \tag{4.27}$$

$$S_{WZ} = -2\pi\alpha' i\, T_{D5} \int F \wedge P[C_{(4)}]$$

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$$= \frac{4iN}{3} \int d\zeta\, F_{\psi\zeta} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8} \sin 4\theta \right) \tag{4.28}$$

where we have used $T_{D5} = N\sqrt{\lambda}/8\pi^4$ and $\mathrm{vol}(\Omega_4) = 8\pi^2/3$. The equation of motion for the electric field states that the conjugate momentum is a constant equal to the number of fundamental string charge k dissolved in the D5 brane

$$\Pi \equiv \frac{-i}{2\pi} \frac{\delta \mathcal{L}}{\delta F_{\psi\zeta}} = \frac{2N}{3\pi} \frac{E \sin^4\theta}{\sqrt{1-E^2}} + \frac{2N}{3\pi} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta \right) = k \tag{4.29}$$

where for convenience we have defined $E = \frac{-2\pi i}{\sqrt{\lambda}} \frac{F_{\psi\zeta}}{\sinh\zeta}$. This equation allows to determine the angle θ at which the D5 sits as a function of k

$$\theta_k - \sin\theta_k \cos\theta_k = \pi\, \frac{k}{N} \tag{4.30}$$

while the electric field is given by $E = \cos\theta_k$. One can check that with this ansatz the equation of motion for u is also satisfied. Adding the appropriate boundary terms for the electric field and the worldvolume scalars (see [13, 30] for details) the on-shell action for the D5 brane becomes

$$S_{D5} = S_{DBI} + S_{WZ} + S_{\mathrm{bdy}} = -\frac{2N}{3\pi} \sqrt{\lambda} \sin^3\theta_k \tag{4.31}$$

so the expectation value of the Wilson loop in the rank k antisymmetric representation is given by

$$\langle W_{A_k} \rangle = \exp\left(\frac{2N}{3\pi} \sqrt{\lambda} \sin^3\theta_k \right) . \tag{4.32}$$

As previously noted in the literature, this result is consistent with the duality between the rank k and rank $N-k$ antisymmetric representations: indeed, it can be seen from eq. (4.30) that under $k \to N-k$ the angle θ_k goes into $\pi - \theta_k$. It can also be checked that in the limit $k/N \to 0$, in which the S^4 factor shrink to zero size, (4.32) coincides with the action of k fundamental strings, as for small k/N eq. (4.30) gives $\theta_k^3 \sim 3\pi k/2N$, so that $\langle W_{A_k} \rangle \simeq \exp k\sqrt{\lambda}$.

4.2.1 Coupling to chiral primaries

The coupling of the KK scalars s^I to the D5 worldvolume can be obtained along the same lines of the D3 calculation of the previous section. However, besides the fluctuation of the AdS_5 part of the metric $h_{\mu\nu}$, we also need the fluctuation of the metric in the S^5 direction $h_{\alpha\beta}$ as well as the fluctuation of the 4-form $a_{(4)}$ along the S^4. The explicit expressions can be found in section 2. In particular, in this coordinates the 4-form over the S^5 is

$$a_{\sigma_1\sigma_2\sigma_3\sigma_4} = 4 \sin^4\theta\, \mu(\Omega_4) \sum s^I\, \partial_\theta Y^I \tag{4.33}$$

where $\sigma_1, \ldots, \sigma_4$ are the coordinates on the S^4 and $\mu(\Omega_4) = \sin^3\sigma_1 \sin^2\sigma_2 \sin\sigma_3$ is the corresponding measure. Differently from the $D3$ compuation, in this case the S^5 spherical harmonics Y^I play an active role in the computation since the $D5$ brane extends into the 5-sphere. The explicit form of the harmonics is given in the appendix.

The variation of the DBI part of the action to first order in the fluctuations $h_{\mu\nu}$ and $h_{\alpha\beta}$ reads

$$S^{(1)}_{DBI} = \frac{T_{D5}}{2}\int \sqrt{\det(\gamma_{ab}+2\pi\alpha' F_{ab})}\left(\gamma_{ab}+2\pi\alpha' F_{ab}\right)^{-1}\left(h_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + h_{\alpha\beta}\partial_a X^\alpha \partial_b X^\beta\right).$$

Using the explicit solution reviewed in the previous section, it is easy to compute the matrix $\gamma_{ab} + 2\pi\alpha' F_{ab}$. Plugging in the explicit expressions for the fluctuations and using the fact that on the D5 solution we have $z = a/\cosh\zeta$ (this follows from the change of coordinate (4.24) after setting $u = 0$), we get after some computations

$$S^{(1)}_{DBI} = \pi T_{D5}\int d\zeta d\sigma_1 \dots d\sigma_4 \mu(\Omega_4)\sinh\zeta \sin^5\theta_k \left(-\frac{4\Delta}{\cosh^2\zeta\sin^2\theta_k} + 8\Delta\right) s^I Y^I. \quad (4.34)$$

Performing the integration over the S^4, only the SO(5) invariant spherical harmonics are selected, namely the harmonics which depends on θ_k only, and we get

$$S^{(1)}_{DBI} = \frac{N\sqrt{\lambda}}{3\pi}\int d\zeta \sinh\zeta \sin^5\theta_k \left(-\frac{4\Delta}{\cosh^2\zeta\sin^2\theta_k} + 8\Delta\right) s^\Delta Y^{\Delta,0}(\theta_k) \quad (4.35)$$

where the suffix on the harmonic indicates that all the quantum numbers except one were set to zero by the integration over the 4-sphere. As reviewed in the appendix, these S^4 invariant harmonics can be explicitely written as

$$Y^{\Delta,0}(\theta_k) = \mathcal{N}_\Delta\, C^{(2)}_\Delta(\cos\theta_k) \quad (4.36)$$

where $C^{(2)}_\Delta(\cos\theta_k)$ are Gegenbauer polynomials, and $\mathcal{N}_\Delta$ is a normalization constant necessary to have orthonormality.

The linear coupling coming from the Wess-Zumino part of the action (4.28) can be obtained using the expression for the 4-form fluctuation in eq. (4.33), and after integrating over the S^4 as above, we get

$$S^{(1)}_{WZ} = \frac{8N\sqrt{\lambda}}{3\pi}\int d\zeta \sinh\zeta \sin^4\theta_k \cos\theta_k s^\Delta\, \partial_{\theta_k} Y^{\Delta,0}(\theta_k). \quad (4.37)$$

4.2.2 The correlation function

The correlator between the rank k antisymmetric Wilson loop and chiral primary operator $\mathcal{O}_\Delta(L)$ can now be computed plugging (3.5) into (4.34) and (4.37) and differentiating with respect to the source s^Δ_0. As before, the bulk-to-boundary propagator can be approximated by $c\, z^\Delta/L^{2\Delta}$. Recalling that on the D5 solution $z = a/\cosh\zeta$, the ζ-integrals can be readily computed and we get

$$\frac{\langle W_{A_k}\,\mathcal{O}_\Delta(L)\rangle}{\langle W_{A_k}\rangle} = \frac{a^\Delta}{L^{2\Delta}}\Bigg[\frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k Y^{\Delta,0}(\theta_k) - \\ -\frac{2^{\Delta/2+1}\sqrt{\lambda}(\Delta+1)}{3\pi\sqrt{\Delta}(\Delta-1)}\sin^5\theta_k\left(\Delta Y^{\Delta,0}(\theta_k) + \frac{\cos\theta_k}{\sin\theta_k}\partial_{\theta_k} Y^{\Delta,0}(\theta_k)\right)\Bigg]. \quad (4.38)$$

Using the formula for the derivatives of Gegenbauer polynomials eq. (A.16), we obtain

$$\Delta Y^{\Delta,0}(\theta_k) + \frac{\cos\theta_k}{\sin\theta_k}\partial_{\theta_k} Y^{\Delta,0}(\theta_k) = \frac{\mathcal{N}_\Delta}{\sin^2\theta_k}\big(\Delta C^{(2)}_\Delta(\cos\theta_k) - (\Delta+3)\cos\theta_k\, C^{(2)}_{\Delta-1}(\cos\theta_k)\big). \tag{4.39}$$

The correlation function (4.38) can then be written as

$$\begin{aligned}\frac{\langle W_{A_k}\,\mathcal{O}_\Delta(L)\rangle}{\langle W_{A_k}\rangle} &= \frac{a^\Delta}{L^{2\Delta}} Y^{\Delta,0}(0)\Bigg[\frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k \cdot \\ &\quad\cdot\frac{6(\Delta-2)!}{(\Delta+2)!}\Big(2(\Delta+1)\cos\theta_k\, C^{(2)}_{\Delta-1}(\cos\theta_k) - \Delta C^{(2)}_\Delta(\cos\theta_k)\Big)\Bigg]\end{aligned} \tag{4.40}$$

where we have factorized out the spherical harmonic evaluated at $\theta = 0$, $Y^{\Delta,0}(0) = \mathcal{N}_\Delta\frac{(\Delta+3)!}{6\Delta!}$.[10] The OPE coefficient $c_{A_k,\Delta}$ we aim to compute is the expression in square brackets. Using the recurrence relation eq. (A.17), we find that this expression can be written in the compact form

$$c_{A_k,\Delta} = \frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k\,\frac{6(\Delta-2)!}{(\Delta+1)!}\,C^{(2)}_{\Delta-2}(\cos\theta_k). \tag{4.41}$$

This is our final result for the correlation function of rank k antisymmetric Wilson loops and chiral primaries. In the next section, we will see that this result exactly matches the one obtained from the normal matrix model. As a check, one can verify that this expression reduces to the string result of [26] in the limit $k/N \to 0$, by using $\theta_k^3 \sim 3\pi k/2N$ and eq. (A.18) from the appendix.

5. The correlation functions from the normal matrix model

It is well-known that the expectation value of a circular Wilson loop in the fundamental representation of SU(N) can be computed from a quadratic Hermitian matrix model [22, 23]

$$\langle W_\Box\rangle = \frac{1}{\mathcal{Z}_H}\int [dM]\exp\left(-\frac{2N}{\lambda}\mathrm{Tr}M^2\right)\frac{1}{N}\mathrm{Tr}_\Box\, e^M. \tag{5.1}$$

This matrix model is conjectured to capture the physics of the Wilson loop exactly, up to instanton corrections [31], to all orders of $1/N$ and λ. The conjecture extends to higher rank Wilson loops as well. The result for the multiply wound Wilson loop has been obtained in [12], whereas [25] and [13, 25] contain the computations for, respectively, the symmetric and antisymmetric representations.

When computing the correlation function between a Wilson loop and a chiral primary operator one can substitute the Hermitian model (5.1) with a complex one by introducing

[10] The OPE coefficient does not include a factor coming from the spherical harmonic evaluated at the unit 6-vector θ^I appearing in eq. (1.1). After a rotation, this vector can always be set to $\theta^I = (1, 0, \ldots, 0)$ which corresponds to the north pole of S^5, *i.e.* $\theta = 0$.

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a second matrix $M_{\rm Im}$ and defining $z = M + iM_{\rm Im}$. In [24] it was shown that, for certain representations of the Wilson loop (the multi-winding and the antisymmetric), the complex matrix model is equivalent to a normal matrix model, which is a complex model where the matrix is constrained to commute with its conjugate. In the normal matrix model the expression for the Wilson loop reads

$$\langle W_{\mathcal{R}} \rangle = \frac{1}{\mathcal{Z}_N} \int_{[z,\bar{z}]=0} [d^2 z] \exp\left(-\frac{2N}{\lambda}\mathrm{Tr} z\bar{z}\right) \frac{1}{\mathrm{dim}\mathcal{R}} \mathrm{Tr}_{\mathcal{R}} e^{\frac{1}{\sqrt{2}}(z+\bar{z})}. \tag{5.2}$$

For large N, the eigenvalues of this model are distributed in incompressible droplets in the complex plane. This leads to interpreting the complex plane as the phase space of free fermions, in analogy with the matrix quantum mechanics describing chiral primary operators [32, 33]. For example, the Wilson loop in the fundamental representation has an eigenvalue distribution given by a circular droplet with constant density [11]

$$\rho(z) = \begin{cases} \frac{2}{\pi\lambda} & |z| < \sqrt{\frac{\lambda}{2}} \\ 0 & |z| > \sqrt{\frac{\lambda}{2}} \end{cases} \tag{5.3}$$

In [24] it was also shown that the correlation function between a Wilson loop in the fundamental representation and a chiral primary operator is given by [12]

$$\langle W_\square \mathcal{O}_\Delta \rangle = \frac{2^{\Delta/2}}{\mathcal{Z}_N} \int_{[z,\bar{z}]=0} [d^2 z] \exp\left(-\mathrm{Tr}\, z\bar{z}\right) \frac{1}{N} \mathrm{Tr}_\square\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z}) - \frac{\lambda}{8N}} \frac{1}{\sqrt{\Delta N^\Delta}} \mathrm{Tr}\, z^\Delta. \tag{5.4}$$

We now use the normal matrix model to check our results for the coefficients of the operator product expansion of higher rank Wilson loops.

5.1 The symmetric case

We start by reproducing the result (4.20) for $c_{S_k,\Delta}$ using the normal matrix model. According to the holographic dictionary put forward in [14], we are interested in the correlator between a Wilson loop in the rank k symmetric representation and the chiral primary operator $\mathcal{O}_\Delta = \frac{1}{\sqrt{\Delta N^\Delta}} \mathrm{Tr} Z^\Delta$. In the limit of large N and large λ the symmetric representation Wilson loop W_{S_k} effectively coincides with the multiply wound fundamental loop $W_\square^{(k)}$, as was shown in [24] and [25]. Therefore we limit ourselves to the simpler case of computing $\langle W_\square^{(k)} \mathcal{O}_\Delta \rangle$, where k is the winding number and corresponds in the brane probe picture to the number of fundamental strings dissolved in the brane.

We start from eq. (4.7) of [24], where we replace everywhere $\lambda \to k^2\lambda$

$$\langle W_\square^{(k)} \mathcal{O}_\Delta \rangle = \frac{2^{\Delta/2+1} e^{k^2\lambda/8N}}{k\sqrt{\Delta\lambda}} \oint \frac{dw}{2\pi i} w^\Delta e^{k\sqrt{\lambda}\, w/2} \left(1 + \frac{k\sqrt{\lambda}}{2Nw}\right)^N \left[\left(1 + \frac{k\sqrt{\lambda}}{2Nw}\right)^\Delta - 1\right] \tag{5.5}$$

[11] Projecting (5.3) into the real axis one recovers the Wigner semi-circle distribution.

[12] The factor $2^{\Delta/2}$ instead of the $2^{-\Delta/2}$ of [24] is set in order to have normalizations consistent with [26].

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The large winding limit consists in taking $N \to \infty$ while keeping $\kappa \equiv \frac{k\sqrt{\lambda}}{4N}$ fixed. In this limit the integral can be evaluated around the saddle point of the terms proportional to N and k

$$\partial_w \left(\frac{k\sqrt{\lambda}}{2} w + N \log \left(1 + \frac{k\sqrt{\lambda}}{2Nw} \right) \right) = 0 \tag{5.6}$$

which yields

$$w_\star = \sqrt{1+\kappa^2} - \kappa. \tag{5.7}$$

Inserting w_* in (5.5) and using

$$\sqrt{1+\kappa^2} + \kappa = \exp(\sinh^{-1}\kappa), \qquad \sqrt{1+\kappa^2} - \kappa = \exp(-\sinh^{-1}\kappa) \tag{5.8}$$

it is easy to see that

$$\langle W_\square^{(k)} \mathcal{O}_\Delta \rangle = \frac{2^{\Delta/2}}{2N\kappa\sqrt{\Delta}} 2\sinh(\Delta \sinh^{-1}\kappa) e^{2N(\kappa\sqrt{1+\kappa^2}+\sinh^{-1}\kappa)}. \tag{5.9}$$

To get a properly normalized expression one still needs to divide (5.9) by

$$\langle W_\square^{(k)} \rangle = \frac{1}{2N\kappa} e^{2N(\kappa\sqrt{1+\kappa^2}+\sinh^{-1}\kappa)}. \tag{5.10}$$

The final result coincides with eq. (4.20), which we obtained from the brane picture.

5.2 The antisymmetric case

To compute the OPE coefficients of Wilson loops in the rank k antisymmetric representation, we have to evaluate the following correlator in the normal matrix model [24]

$$\langle W_{A_k} \mathcal{O}_\Delta \rangle = \frac{2^{\Delta/2}\, e^{k\lambda/8N}}{\mathcal{Z}_N N^{\Delta/2}\sqrt{\Delta}} \int_{[z,\bar{z}]=0} [d^2z] e^{-\mathrm{Tr}(z\bar{z})} \frac{1}{\dim A_k} \mathrm{Tr}_{A_k} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\, \mathrm{Tr} z^\Delta. \tag{5.11}$$

This matrix integral can actually be solved exactly, as was shown in [24], and similarly to the case of the fundamental representation, it can be written as a k-dimensional contour integral. However, it does not seem to be easy to take the large N and large k limit with k/N fixed from such an expression. Here we follow a different approach to get the above correlator in this limit. First, as in [25], we find it convenient to rewrite the trace in the antisymmetric representation using the corresponding generating function

$$\mathrm{Tr}_{A_k} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})} = \oint \frac{dt}{2\pi i} t^{k-1} \exp\left[\mathrm{Tr} \log \left(1 + \frac{1}{t} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})} \right) \right]. \tag{5.12}$$

Since we expect the correlator to be real, it is also convenient to replace $\mathrm{Tr} z^\Delta \to \frac{1}{2}(\mathrm{Tr} z^\Delta + \mathrm{Tr}\bar{z}^\Delta)$. The idea is then to view the insertion of the chiral primary $\mathrm{Tr} z^\Delta$ in (5.11) as a

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small perturbation of the gaussian potential, by writing

$$\int_{[z,\bar z]=0}[d^2z]e^{-\mathrm{Tr}(z\bar z)}\mathrm{Tr}z^\Delta\, e^{\mathrm{Tr}\log(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar z)})}=$$

$$=\mathcal{Z}_N\frac{\partial}{\partial\alpha}\left(\frac{1}{\mathcal{Z}_N(\alpha)}\int_{[z,\bar z]=0}[d^2z]e^{-\mathrm{Tr}(z\bar z)+\frac{\alpha}{2}(\mathrm{Tr}z^\Delta+\mathrm{Tr}\bar z^\Delta)}e^{\mathrm{Tr}\log(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar z)})}\right)\Bigg|_{\alpha=0}$$

$$\equiv\mathcal{Z}_N\frac{\partial}{\partial\alpha}\left\langle\exp\left[\mathrm{Tr}\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar z)}\right)\right]\right\rangle_\alpha\Bigg|_{\alpha=0} \tag{5.13}$$

where we have introduced an α-dependent partition function

$$\mathcal{Z}_N(\alpha)=\int_{[z,\bar z]=0}[d^2z]e^{-\mathrm{Tr}(z\bar z)+\frac{\alpha}{2}(\mathrm{Tr}z^\Delta+\mathrm{Tr}\bar z^\Delta)} \tag{5.14}$$

and we have used that $\mathcal{Z}_N(\alpha)=\mathcal{Z}_N+\mathcal{O}(\alpha^2)$.[13] The problem is now to evaluate the correlation function (5.13) in the normal matrix model with the deformed potential

$$V(z,\bar z)=-\mathrm{Tr}z\bar z+\frac{\alpha}{2}\mathrm{Tr}(z^\Delta+\bar z^\Delta). \tag{5.15}$$

Normal models with potentials of this kind were previously studied in the literature, for a recent account see for example [34, 35]. To solve the model at large N, one can as usual go to the eigenvalue basis at the expenses of introducing a Vandermonde factor, and determine the eigenvalue density $\rho_\alpha(z,\bar z)$ in the continuum limit. The density is found by solving the saddle point equation [14]

$$z-\frac{\Delta\alpha}{2}\bar z^{\Delta-1}=N\int d^2z'\frac{\rho_\alpha(z',\bar z')}{\bar z-\bar z'} \tag{5.16}$$

where the term in the right hand side comes from the Vandermonde factor. Once the density is known, the correlation function in (5.13) becomes

$$\left\langle\exp\left[\mathrm{Tr}\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar z)}\right)\right]\right\rangle_\alpha\to\exp\left[N\int d^2z\,\rho_\alpha(z,\bar z)\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar z)}\right)\right]. \tag{5.17}$$

It is known that for potentials of the kind $V(z,\bar z)=-z\bar z+f(z)+\bar f(\bar z)$, the density is a constant (equal to $\frac{1}{N\pi}$ in the normalizations we are using here) inside a certain droplet in the complex plane and zero outside. For the gaussian potential, as reviewed previously, the droplet is just a circle of radius $\sqrt{N}$ (to compare with eq. (5.3), one has to rescale $z\to\sqrt{\frac{\lambda}{2N}}z$), while the term proportional to α induces a deformation of the circle which preserves its total area (since we do not change the number of eigenvalues). It is not difficult to find the shape of the droplet which solves the saddle point equation (5.16) at

[13] This follows from the fact that in the matrix model with gaussian potential $\langle\mathrm{Tr}z^\Delta\rangle=\langle\mathrm{Tr}\bar z^\Delta\rangle=0$.

[14] As in [25], the term $\exp\left[\mathrm{Tr}\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar z)}\right)\right]$ does not modify the saddle point equation at leading order at large N.

leading order in α. It is convenient to work in polar coordinates $z = re^{i\phi}$. The curve which bounds the droplet can then be written at first order as

$$r(\phi) = \sqrt{N}\big(1 + \alpha\, f(\phi)\big). \tag{5.18}$$

Clearly $f(\phi)$ has to be periodic, and may be written as

$$f(\phi) = \sum_{n=1}^{\infty} a_n \cos n\phi \tag{5.19}$$

where only cosines appear because of the symmetry of the potential (5.15) under $z \leftrightarrow \bar{z}$, and the mode with $n = 0$ is excluded by requiring the area to be preserved. The saddle point equation now reads

$$r\, e^{i\phi} - \frac{\Delta\alpha}{2} r^{\Delta-1}\, e^{-i(\Delta-1)\phi} = \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_0^{\sqrt{N}(1+\alpha\, f(\phi'))} dr'\, \frac{r'}{r\, e^{-i\phi} - r'\, e^{-i\phi'}}. \tag{5.20}$$

Expanding the integral at first order in α and plugging in the Fourier expansion (5.19), we see that this equation is satisfied if $a_\Delta = N^{\Delta/2-1}\,\frac{\Delta}{2}$ and all other a_n vanish, so we find that the shape of the deformed droplet is given by the curve

$$r(\phi) = \sqrt{N}\left(1 + \frac{\alpha}{2}\,\Delta\, N^{\Delta/2-1} \cos\Delta\phi\right). \tag{5.21}$$

Before moving on to compute (5.13), we can check the validity of the method by applying it to the computation of the correlator (5.4) when the Wilson loop is in the fundamental representation. In this case, following the same steps as above, in the large N limit we arrive at

$$\begin{aligned}
\langle W_\square \mathcal{O}_\Delta\rangle &= \frac{2^{\Delta/2}}{\sqrt{\Delta}N^{\Delta/2}}\frac{\partial}{\partial\alpha}\int d^2z\, \rho_\alpha(z,\bar{z})\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\Big|_{\alpha=0} \\
&= \frac{2^{\Delta/2}}{\sqrt{\Delta}N^{\Delta/2}}\frac{\partial}{\partial\alpha}\frac{1}{N\pi}\int_0^{2\pi} d\phi \int_0^{\sqrt{N}(1+\frac{\alpha}{2}\,\Delta\, N^{\Delta/2-1}\cos\Delta\phi)} dr\, r\, e^{\sqrt{\frac{\lambda}{N}}r\cos\phi}\Big|_{\alpha=0} \\
&= \frac{2^{\Delta/2}\sqrt{\Delta}}{N}\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{\sqrt{\lambda}\cos\phi}\cos\Delta\phi = \frac{2^{\Delta/2}\sqrt{\Delta}}{N} I_\Delta(\sqrt{\lambda})
\end{aligned} \tag{5.22}$$

which is the result first found in [36] and the correct large N limit of the exact formula (5.5) (with $k = 1$).

Going back to the antisymmetric representation, we have to evaluate

$$\begin{aligned}
&\oint \frac{dt}{2\pi i} t^{k-1} \frac{\partial}{\partial\alpha} \exp\left[N\int d^2z\, \rho_\alpha(z,\bar{z}) \log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]\Big|_{\alpha=0} \\
&= \oint \frac{dt}{2\pi i} t^{k-1} \frac{\partial}{\partial\alpha}\left[N\int d^2z\, \rho_\alpha(z,\bar{z}) \log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]\Big|_{\alpha=0} \times \\
&\qquad \times \exp\left[N\int d^2z\, \rho_0(z,\bar{z}) \log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]
\end{aligned} \tag{5.23}$$

where in the last line ρ_0 is just the circular droplet density. Since the exponent is independent of α, the t integral can be evaluated in the supergravity limit of large λ exactly as in [25]: we first make a change of variables $t = e^{\sqrt{\lambda} w}$, then the saddle point of the exponent is found to be

$$w_\star = \cos\theta_k \tag{5.24}$$

where θ_k is defined as in eq. (4.30). The exponent in (5.23) gives a term proportional to the expectation value of the Wilson loop, while the prefactor is evaluated at the saddle point. After dividing by $\langle W_{A_k}\rangle$, the OPE coefficient can then be obtained as

$$\begin{aligned}
\frac{\langle W_{A_k}\,\mathcal{O}_\Delta\rangle}{\langle W_{A_k}\rangle} &= \frac{2^{\Delta/2}}{\sqrt{\Delta}N^{\Delta/2}}\frac{\partial}{\partial\alpha}\left[N\int d^2z\,\rho_\alpha(z,\bar z)\log\left(1+e^{\frac{1}{2}\sqrt{\lambda}\left(\frac{z+\bar z}{\sqrt{N}}-2\cos\theta_k\right)}\right)\right]\Bigg|_{\alpha=0} \\
&\simeq \frac{2^{\Delta/2}N\sqrt{\lambda}}{\sqrt{\Delta}N^{\Delta/2}}\frac{\partial}{\partial\alpha}\frac{2}{\pi}\int_0^{\theta_k}d\phi\int_{\frac{\cos\theta_k}{\cos\phi}}^{1+\frac{\alpha}{2}\Delta N^{\Delta/2-1}\cos\Delta\phi}dr\,r(r\cos\phi-\cos\theta_k)\Big|_{\alpha=0}
\end{aligned} \tag{5.25}$$

where the lower limit in the r integral comes from the fact that in the large λ limit the integral has support only in the region $r\cos\phi \geq \cos\theta_k$.[15] After doing the derivative, (5.25) gives the final result

$$\frac{\langle W_{A_k}\,\mathcal{O}_\Delta\rangle}{\langle W_{A_k}\rangle} = \frac{2^{\Delta/2}\sqrt{\Delta\lambda}}{\pi}\int_0^{\theta_k}d\phi\,\cos\Delta\phi\,(\cos\phi-\cos\theta_k). \tag{5.26}$$

Remarkably, the integral in (5.26) precisely reproduces the Gegenbauer polynomials arising in the bulk computation, and the final result is

$$\frac{\langle W_{A_k}\,\mathcal{O}_\Delta\rangle}{\langle W_{A_k}\rangle} = \frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k\,\frac{6(\Delta-2)!}{(\Delta+1)!}\,C^{(2)}_{\Delta-2}(\cos\theta_k) \tag{5.27}$$

which exactly matches the $D5$ computation of the OPE coefficient.

6. Conclusion

In this paper we computed the correlation function between a higher rank Wilson loop and a chiral primary operator in the fundamental representation using branes with electric fluxes. Following the proposal of [12 – 14], we considered a D3_k brane for the rank k symmetric case and a D5_k brane for the antisymmetric one. We then checked our results with the normal matrix model discussed in [24] finding perfect agreement in both cases.

We focussed on chiral primary operators but it should not be difficult to extend our computation to operators corresponding to other supergravity modes. For example, the KK modes of the dilaton are necessary to compute correlation functions of Wilson loops and $\mathrm{Tr}\,\Phi^\Delta F_+^2$.

[15] The upper limit in the ϕ integral is rigorously $\theta_k + \mathcal{O}(\alpha)$, but it is easy to see that the correction does not contribute at first order in α.

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It would be worthwhile to study more general representations of both the Wilson loop and the chiral primary operator. A particularly interesting issue to address would be understanding from our brane picture the selection rule found in [24]: for Wilson loops in the rank k antisymmetric representation the only non vanishing correlators involve chiral primaries with traces over Young diagrams with at most k hooks. Another direction to pursue may be considering the correlation function between higher dimensional Wilson loop and a chiral primary operator with $\Delta \sim N$. In the bulk this would require to study the bulk-to-bulk exchange of supergravity degrees of freedom between the electric branes describing the Wilson loop and the (dual) giant gravitons associated with the chiral primary.

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A. Spherical harmonics and orthogonal polynomials

In this appendix we collect some facts about spherical harmonics and orthogonal polynomials we have used in the paper. We follow the treatments of [37] and [38].

Spherical harmonics in d dimensions are eigenfunctions of the Laplacian on the unit d-sphere

$$\nabla^2_{(d)} Y^I(\Omega) = \lambda Y^I(\Omega) \tag{A.1}$$

where the Laplacian is

$$\nabla^2_{(d)} = \frac{1}{\sqrt{\det g}}\, \partial_i \sqrt{\det g}\, g^{ij}\, \partial_j \tag{A.2}$$

with the metric given by $g_{ij} = \mathrm{diag}(1, \sin^2\theta_d(1, \sin^2\theta_{d-1}(\ldots)))$. The integer multi-index $I = (l_d, \ldots, l_1)$ satisfies

$$l_d \geq l_{d-1} \geq \cdots \geq l_2 \geq |l_1|. \tag{A.3}$$

The general solution to eq. (A.1) is

$$Y^{l_d,\ldots,l_1}(\theta_d, \ldots, \theta_1) = \frac{e^{i l_1 \theta_1}}{\sqrt{2\pi}} \prod_{n=2}^{d} {}_n\bar{P}^{\,l_{n-1}}_{l_n}(\theta_n) \tag{A.4}$$

where we defined

$${}_n\bar{P}^{\,l}_{L}(\theta) = {}_n c^{\,l}_{L} (\sin\theta)^{-(n-2)/2} P^{-(l+(n-2)/2)}_{L+(n-2)/2}(\cos\theta). \tag{A.5}$$

In this expression $P_n^{-m}(x)$ is the Legendre function of the first kind and the constant

$$ {}_n c_L^l = \left[\frac{(2L+n-1)(L+l+n-2)!}{2(L-l)!}\right]^{1/2} \tag{A.6} $$

is chosen to ensure the orthonormalization condition

$$ \int \mu(\Omega_d)\; Y^I Y^{I'} = \delta^{II'} \tag{A.7} $$

where $\mu(\Omega_d)$ is the measure over S^d. The integration over S^{d-1} selects only SO(d) invariant harmonics

$$ \int \mu(\Omega_{d-1}) \sum_I Y^I = \sum_{l_d} Y^{l_d,\,0,\ldots,0}. \tag{A.8} $$

The eigenvalue λ depends only on $l_d \equiv \Delta$ because of the $O(d+1)$ symmetry of the problem and it can be found by studying the action of the Laplacian on SO($d-1$) invariant spherical harmonics

$$ \left(\frac{1}{\sin^{d-1}\theta_d}\frac{\partial}{\partial\theta_d}\,\sin^{d-1}\theta_d\,\frac{\partial}{\partial\theta_d}\right) Y^{\Delta,0}(\Omega) = \lambda_\Delta Y^{\Delta,0}(\Omega). \tag{A.9} $$

After the change of variable $x = \cos\theta_d$, this is recognized to be the Gegenbauer equation

$$ \left((1-x^2)\frac{\partial^2}{\partial x^2} - d\,x\frac{\partial}{\partial x}\right) Y^{\Delta,0}(x) = \lambda_\Delta Y^{\Delta,0}(x). \tag{A.10} $$

The solution to this equation is

$$ \lambda_\Delta = -\Delta(\Delta+d-1)\,, \qquad\qquad Y^{\Delta,0}(x) = \mathcal{N}_\Delta\, C_\Delta^{\left(\frac{d-1}{2}\right)}(x) \tag{A.11} $$

where $C_\Delta^{\left(\frac{d-1}{2}\right)}$ are Gegenbauer polynomials and the constant $\mathcal{N}_\Delta$ can be obtained from the orthonormality of the $Y^{\Delta,0}$'s

$$ \mathcal{N}_\Delta = \left[\frac{\Delta!(2\Delta+d-1)\left[\Gamma\left(\frac{d-1}{2}\right)\right]^2\Gamma\left(\frac{d}{2}\right)}{2^{4-d}\pi^{\frac{d+2}{2}}\Gamma(\Delta+d-1)}\right]^{1/2}. \tag{A.12} $$

The Gegenbauer polynomials $C_\Delta^{(\lambda)}(x)$ are a generalization of the Legendre polynomials and can be obtained from the following generating function

$$ \frac{1}{(1-2xt+t^2)^\lambda} = \sum_{\Delta=0}^{\infty} C_\Delta^{(\lambda)}(x)\,t^\Delta. \tag{A.13} $$

We list the first few of them

$$ \begin{aligned} &C_0^{(\lambda)}(x) = 1 \\ &C_1^{(\lambda)}(x) = 2\lambda x \\ &C_2^{(\lambda)}(x) = -\lambda + 2\lambda(1+\lambda)x^2 \end{aligned} $$

$$C_3^{(\lambda)}(x) = -2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3. \tag{A.14}$$

They satisfy the normalization condition

$$\int_{-1}^{1} dx(1-x^2)^{\lambda-1/2}\left[C_\Delta^{(\lambda)}\right]^2 = 2^{1-2\lambda}\pi\frac{\Gamma(\Delta+2\lambda)}{(\Delta+\lambda)\Gamma^2(\lambda)\Gamma(\Delta+1)} \tag{A.15}$$

for $\lambda > -1/2$.

In the paper we have used the following formula for the derivative of a Gegenbauer polynomial

$$(1-x^2)\partial_x C_\Delta^{(\lambda)}(x) = -\Delta\, x\, C_\Delta^{(\lambda)}(x) + (\Delta+2\lambda-1)C_{\Delta-1}^{(\lambda)}(x) \tag{A.16}$$

and the following recurrence relation

$$\Delta\, C_\Delta^{(\lambda)}(x) = 2(\Delta+\lambda-1)x\, C_{\Delta-1}^{(\lambda)}(x) - (\Delta+2\lambda-2)C_{\Delta-2}^{(\lambda)}(x). \tag{A.17}$$

We have also used that

$$C_\Delta^{\left(\frac{d-1}{2}\right)}(1) = \frac{(\Delta+d-2)!}{\Delta!(d-2)!}. \tag{A.18}$$

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